

FEEDBACK STABILIZATION OVER COMMUTATIVE RINGS: FURTHER STUDY OF THE COORDINATE-FREE APPROACH

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Abstract. This paper is concerned with the coordinate-free approach to control systems. The coordinate-free approach is a factorization approach but does not require the coprime factorizations of the plant. We present two criteria for feedback stabilizability for MIMO systems in which transfer functions belong to the total rings of fractions of commutative rings. Both of them are generalizations of Sule's results in [SIAM J. Control Optim., **32**–6, 1675–1695(1994)]. The first criterion is expressed in terms of modules generated from a causal plant and does not require the plant to be strictly causal. It shows that if the plant is stabilizable, the modules are projective. The other criterion is expressed in terms of ideals called generalized elementary factors. This gives the stabilizability of a causal plant in terms of the coprimeness of the generalized elementary factors. As an example, a discrete finite-time delay system is considered.

Key words. Linear systems, Feedback stabilization, Coprime factorization over commutative rings

AMS subject classifications. 93C05, 93D15, 93B50, 93B25

Abbreviated Title. Feedback Stabilization over Commutative Rings

1. Introduction. In this paper we are concerned with the coordinate-free approach to control systems. This approach is a factorization approach but does not require the coprime factorizations of the plant.

The factorization approach to control systems has the advantage that it embraces, within a single framework, numerous linear systems such as continuous-time as well as discrete-

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time systems, lumped as well as distributed systems, 1-D as well as n -D systems, etc.[14]. This factorization approach was patterned after Desoer *et al.*[3] and Vidyasagar *et al.*[14]. In this approach, when problems such as feedback stabilization are studied, one can focus on the key aspects of the problem under study rather than be distracted by the special features of a particular class of linear systems. A transfer function of this approach is given as the ratio of two stable causal transfer functions and the set of stable causal transfer functions forms a commutative ring. For a long time, the theory of the factorization approach had been founded on the coprime factorizability of transfer matrices, which is satisfied in the case where the set of stable causal transfer functions is such a commutative ring as a Euclidean domain, a principal ideal, or a Bézout domain. However, Anantharam in [1] showed that there exist models in which some stabilizable plants do not have right-/left-coprime factorizations.

Recently, Shankar and Sule in [10] have presented a theory of feedback stabilization for single-input single-output (SISO) transfer functions having fractions over general integral domains. Moreover, Sule in [11, 12] has presented a theory of the feedback stabilization of strictly causal plants for multi-input multi-output (MIMO) transfer matrices, in which transfer functions belong to the total rings of fractions of commutative rings, with some restrictions. Their approach to the control systems is called a “coordinate-free approach”([10, p.15]) in the sense that they do not require the coprime factorizability of transfer matrices.

The main contribution of this paper consists of providing two criteria for feedback stabilizability for MIMO systems in which transfer functions belong to the total rings of fractions of commutative rings: the first criterion is expressed in terms of modules ((ii) of Theorem 3.3) and the other in terms of ideals called generalized elementary factors ((iii) of Theorem 3.3). They are more general than Sule’s results in the following sense: (i) our results do not require that plants are strictly causal; (ii) we do not employ the restriction of commutative

rings. Further, we will not use the theory of algebraic geometry.

The paper is organized as follows. In § 2, we give mathematical preliminaries, set up the feedback stabilization problem, present the previous results, and define the causality of the transfer functions. In § 3, we state our main results. As a preface to our main results, we also introduce there the notion of the generalized elementary factor of a plant. In § 4, we give intermediate results which we will utilize in the proof of the main theorem. In § 5, we prove our main theorem. In § 6, we discuss the causality of the stabilizing controllers. Also, in order to make the contents clear, we present examples concerning a discrete finite-time delay system in § 3, 4, 5 in series.

2. Preliminaries. In the following we begin by introducing the notations of commutative rings, matrices, and modules, commonly used in this paper. Then we give in order the formulation of the feedback stabilization problem, the previous results, and the causality of transfer functions.

2.1. Notations.

Commutative Rings. In this paper, we consider that any commutative ring has the identity 1 different from zero. Let \mathcal{R} denote a commutative ring. A *zerodivisor* in \mathcal{R} is an element x for which there exists a nonzero y such that $xy = 0$. In particular, a zerodivisor x is said to be *nilpotent* if $x^n = 0$ for some positive integer n . The set of all nilpotent elements in \mathcal{R} , which is an ideal, is called the *nilradical* of \mathcal{R} . A *nonzerodivisor* in \mathcal{R} is an element which is not a zerodivisor. The total ring of fractions of \mathcal{R} is denoted by $\mathcal{F}(\mathcal{R})$.

The set of all prime ideals of \mathcal{R} is called the *prime spectrum* of \mathcal{R} and is denoted by $\text{Spec } \mathcal{R}$. The prime spectrum of \mathcal{R} is said to be *irreducible* as a topological space if every non-empty open set is dense in $\text{Spec } \mathcal{R}$.

We will consider that the set of stable causal transfer functions is a commutative ring, denoted by \mathcal{A} . Further, we will use the following three kinds of rings of fractions:

(i) The first one appears as the total ring of fractions of \mathcal{A} , which is denoted by $\mathcal{F}(\mathcal{A})$ or simply by \mathcal{F} ; that is, $\mathcal{F} = \{n/d \mid n, d \in \mathcal{A}, d \text{ is a nonzerodivisor}\}$. This will be considered to be the set of all possible transfer functions. If the commutative ring \mathcal{A} is an integral domain, \mathcal{F} becomes a field of fractions of \mathcal{A} . However, if \mathcal{A} is not an integral domain, then \mathcal{F} is not a field, because any zerodivisor of \mathcal{F} is not a unit.

(ii) The second one is associated with the set of powers of a nonzero element of \mathcal{A} . Suppose that f denotes a nonzero element of \mathcal{A} . Given a set $S_f = \{1, f, f^2, \dots\}$, which is a multiplicative subset of \mathcal{A} , we denote by \mathcal{A}_f the ring of fractions of \mathcal{A} with respect to the multiplicative subset S_f ; that is, $\mathcal{A}_f = \{n/d \mid n \in \mathcal{A}, d \in S_f\}$. We point out two facts here: (a) In the case where f is nilpotent, \mathcal{A}_f becomes isomorphic to $\{0\}$. (b) In the case where f is a zerodivisor, even if the equality $a/1 = b/1$ holds over \mathcal{A}_f with $a, b \in \mathcal{A}$, we cannot say in general that $a = b$ over \mathcal{A} ; alternatively, $a = b + z$ over \mathcal{A} holds with some zerodivisor z of \mathcal{A} such that $zf^\omega = 0$ with a sufficiently large integer ω .

(iii) The last one is the total ring of fractions of \mathcal{A}_f , which is denoted by $\mathcal{F}(\mathcal{A}_f)$; that is, $\mathcal{F}(\mathcal{A}_f) = \{n/d \mid n, d \in \mathcal{A}_f, d \text{ is a nonzerodivisor of } \mathcal{A}_f\}$. If f is a nonzerodivisor of \mathcal{A} , $\mathcal{F}(\mathcal{A}_f)$ coincides with the total ring of fractions of \mathcal{A} . Otherwise, they may not coincide.

The reader is referred to Chapter 3 of [2] for the ring of fractions and to Chapter 1 of [2] for the prime spectrum.

In the rest of the paper, we will use \mathcal{R} as an unspecified commutative ring and mainly suppose that \mathcal{R} denotes either \mathcal{A} or \mathcal{A}_f .

Matrices. Suppose that x and y denote sizes of matrices.

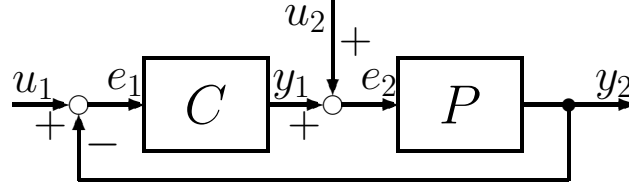
The set of matrices over \mathcal{R} of size $x \times y$ is denoted by $\mathcal{R}^{x \times y}$. In particular, the set of square matrices over \mathcal{R} of size x is denoted by $(\mathcal{R})_x$. A square matrix is called *singular* over \mathcal{R} if its determinant is a zerodivisor of \mathcal{R} , and *nonsingular* otherwise. The identity and the zero matrices are denoted by E_x and $O_{x \times y}$, respectively, if the sizes are required,

otherwise they are denoted simply by E and O . For a matrix A over \mathcal{R} , the inverse matrix of A is denoted by A^{-1} provided that $\det(A)$ is a unit of $\mathcal{F}(\mathcal{R})$. The ideal generated by \mathcal{R} -linear combination of all minors of size m of a matrix A is denoted by $I_{m\mathcal{R}}(A)$.

Matrices A and B over \mathcal{R} are *right-coprime over \mathcal{R}* if there exist matrices \tilde{X} and \tilde{Y} over \mathcal{R} such that $\tilde{X}A + \tilde{Y}B = E$. Analogously, matrices \tilde{A} and \tilde{B} over \mathcal{R} are *left-coprime over \mathcal{R}* if there exist matrices X and Y over \mathcal{R} such that $\tilde{A}X + \tilde{B}Y = E$. Note that, in the sense of the above definition, even if two matrices have no common right-(left-)divisors except invertible matrices, they may not be right-(left-)coprime over \mathcal{R} . (For example, two matrices $[z_1]$ and $[z_2]$ of size 1×1 over the bivariate polynomial ring $\mathbb{R}[z_1, z_2]$ over the real field \mathbb{R} are neither right- nor left-coprime over $\mathbb{R}[z_1, z_2]$ in our setting.) Further, a pair (N, D) of matrices N and D is said to be a *right-coprime factorization of P over \mathcal{R}* if (i) the matrix D is nonsingular over \mathcal{R} , (ii) $P = ND^{-1}$ over $\mathcal{F}(\mathcal{R})$, and (iii) N and D are right-coprime over \mathcal{R} . Also, a pair (\tilde{N}, \tilde{D}) of matrices \tilde{N} and \tilde{D} is said to be a *left-coprime factorization of P over \mathcal{R}* if (i) \tilde{D} is nonsingular over \mathcal{R} , (ii) $P = \tilde{D}^{-1}\tilde{N}$ over $\mathcal{F}(\mathcal{R})$, and (iii) \tilde{N} and \tilde{D} are left-coprime over \mathcal{R} . As we have seen, in the case where a matrix is potentially used to express *left* fractional form and/or *left* coprimeness, we usually attach a tilde ‘ \sim ’ to a symbol; for example \tilde{N}, \tilde{D} for $P = \tilde{D}^{-1}\tilde{N}$ and \tilde{Y}, \tilde{X} for $\tilde{Y}N + \tilde{X}D = E$.

Modules. For a matrix A over \mathcal{R} , we denote by $M_r(A)$ ($M_c(A)$) the \mathcal{R} -module generated by rows (columns) of A .

Suppose that $A, B, \tilde{A}, \tilde{B}$ are matrices over \mathcal{R} and X is a matrix over $\mathcal{F}(\mathcal{R})$ such that $X = AB^{-1} = \tilde{B}^{-1}\tilde{A}$ with B and \tilde{B} being nonsingular. Then the \mathcal{R} -module $M_r([A^t \ B^t]^t)$ ($M_c([\tilde{A} \ \tilde{B}])$) is uniquely determined up to isomorphism with respect to any choice of fractions AB^{-1} ($\tilde{B}^{-1}\tilde{A}$) of X as shown in Lemma 2.1 below. Thus for a matrix X over $\mathcal{F}(\mathcal{R})$, we denote by $\mathcal{T}_{X,\mathcal{R}}$ and $\mathcal{W}_{X,\mathcal{R}}$ the modules $M_r([A^t \ B^t]^t)$ and $M_c([\tilde{A} \ \tilde{B}])$, respectively. If $\mathcal{R} = \mathcal{A}$, we write simply \mathcal{T}_X and \mathcal{W}_X for $\mathcal{T}_{X,\mathcal{A}}$ and $\mathcal{W}_{X,\mathcal{A}}$, respectively. We will use, for

FIG. 2.1. Feedback system Σ .

example, the notations \mathcal{T}_P , \mathcal{W}_P , \mathcal{T}_C , and \mathcal{W}_C for the matrices P and C over \mathcal{F} .

An \mathcal{R} -module M is called *free* if it has a basis, that is, a linearly independent system of generators. The *rank* of a free \mathcal{R} -module M is equal to the cardinality of a basis of M , which is independent of the basis chosen. An \mathcal{R} -module M is called *projective* if it is a direct summand of a free \mathcal{R} -module, that is, there is a module N such that $M \oplus N$ is free. The reader is referred to Chapter 2 of [2] for the module theory.

LEMMA 2.1. *Suppose that X is a matrix over $\mathcal{F}(\mathcal{R})$ and is expressed in the form of a fraction $X = AB^{-1} = \tilde{B}^{-1}\tilde{A}$ with some matrices $A, B, \tilde{A}, \tilde{B}$ over \mathcal{R} . Then the \mathcal{R} -module $M_r([A^t \ B^t]^t) \left(M_c([\tilde{A} \ \tilde{B}]) \right)$ is uniquely determined up to isomorphism with respect to any choice of fractions $AB^{-1} \left(\tilde{B}^{-1}\tilde{A} \right)$ of X .*

Proof. Without loss of generality, it is sufficient to show that $M_r([A_1^t \ b_1 E]^t) \simeq M_r([A_2^t \ B_2^t]^t)$, where $b_1 \in \mathcal{R}$ and $A_1(b_1 E)^{-1} = A_2 B_2^{-1}$. Since b_1 is a nonzerodivisor and B_2 is nonsingular, we have $M_r([A_1^t \ b_1 E]^t) \simeq M_r([A_1^t \ b_1 E]^t B_2) \simeq M_r([A_2^t \ B_2^t]^t b_1 E) \simeq M_r([A_2^t \ B_2^t]^t)$. The other isomorphism can be proved analogously. \square

2.2. Feedback Stabilization Problem. The stabilization problem considered in this paper follows that of Sule in [11] who considers the feedback system Σ [13, Ch.5, Figure 5.1] as in Figure 2.1. For further details the reader is referred to [13]. Let a commutative ring \mathcal{A} represent the set of *stable causal* transfer functions. The total ring of fractions of \mathcal{A} , denoted by \mathcal{F} , consists of *all* possible transfer functions. The set of matrices of size $x \times y$ over \mathcal{A} ,

denoted by $\mathcal{A}^{x \times y}$, coincides with the set of stable causal transfer matrices of size $x \times y$. Also the set of matrices of size $x \times y$ over \mathcal{F} , denoted by $\mathcal{F}^{x \times y}$, coincides with all possible transfer matrices of size $x \times y$. Throughout the paper, the plant we consider has m inputs and n outputs, and its transfer matrix, which itself is also called simply a *plant*, is denoted by P and belongs to $\mathcal{F}^{n \times m}$. We will occasionally consider matrices over $\mathcal{A}(\mathcal{F})$ as ones over \mathcal{A}_f or $\mathcal{F}(\mathcal{A}_f)$ by natural mapping.

DEFINITION 2.2. Define \widehat{F}_{ad} by

$$\widehat{F}_{\text{ad}} = \{(X, Y) \in \mathcal{F}^{x \times y} \times \mathcal{F}^{y \times x} \mid \det(E_x + XY) \text{ is a unit of } \mathcal{F}, \\ x \text{ and } y \text{ are positive integers}\}.$$

For $P \in \mathcal{F}^{n \times m}$ and $C \in \mathcal{F}^{m \times n}$, the matrix $H(P, C) \in (\mathcal{F})_{m+n}$ is defined by

$$(2.1) \quad H(P, C) = \begin{bmatrix} (E_n + PC)^{-1} & -P(E_m + CP)^{-1} \\ C(E_n + PC)^{-1} & (E_m + CP)^{-1} \end{bmatrix}$$

provided $(P, C) \in \widehat{F}_{\text{ad}}$. This $H(P, C)$ is the transfer matrix from $[u_1^t \ u_2^t]^t$ to $[e_1^t \ e_2^t]^t$ of the feedback system Σ . If (i) $(P, C) \in \widehat{F}_{\text{ad}}$ and (ii) $H(P, C) \in (\mathcal{A})_{m+n}$, then we say that the plant P is stabilizable, P is stabilized by C , and C is a stabilizing controller of P .

2.3. Previous Results. In [11] Sule gave the results of the feedback stabilizability. We show them after introducing the notion of the elementary factor which is used to state his results.

DEFINITION 2.3. (Elementary Factors [11, p.1689]) Assume that \mathcal{A} is a unique factorization domain. Denote by T the matrix $[N^t \ dE_m]^t$ and by W the matrix $[N \ dE_n]$ over \mathcal{A} , where $P = Nd^{-1}$ with $N \in \mathcal{A}^{n \times m}$, $d \in \mathcal{A}$. Let $\{T_1, T_2, \dots, T_r\}$ be the family of all nonsingular $m \times m$ submatrices of the matrix T , and for each index i , let f_i be the radical of the least common multiple of all the denominators of TT_i^{-1} . The family $F = \{f_1, f_2, \dots, f_r\}$

is called the family of elementary factors of the matrix T . Analogously let $\{W_1, W_2, \dots, W_r\}$ be the family of all nonsingular $n \times n$ submatrices of the matrix W , and for each index j , let g_j be the radical of the least common multiple of all the denominators of $W_j^{-1}W$. Let $G = \{g_1, g_2, \dots, g_l\}$ denote the family of elementary factors of the transposed matrix W^t . Now let $H = \{f_i g_j \mid i = 1, \dots, r, j = 1, \dots, l\}$. This family H is called the elementary factor of the transfer matrix P .

Then, Sule's two elegant results can be rewritten as follows. The first result assumes that the prime spectrum of \mathcal{A} is irreducible. The second one assumes that \mathcal{A} is a unique factorization domain.

THEOREM 2.4. (Theorem 1 of [11]) *Suppose that the prime spectrum of \mathcal{A} is irreducible. Further suppose that a plant P of size $n \times m$ is strictly causal, where the notion of the strictly causal is defined as in [12] (rather than [11]). Then the plant P is stabilizable if and only if the following conditions are satisfied:*

- (i) *The module \mathcal{T}_P is projective of rank m .*
- (ii) *The module \mathcal{W}_P is projective of rank n . \square*

Recall that for a matrix X over \mathcal{F} , we use the notations \mathcal{T}_X and \mathcal{W}_X to denote \mathcal{A} -modules generated by using the matrix X . Further it should be noted that the definitions of \mathcal{T}_P and \mathcal{W}_P in [11] are slightly different from those of this paper. Nevertheless this is not a problem by virtue of Lemma 2.1.

THEOREM 2.5. (Theorem 4 of [11]) *Suppose that \mathcal{A} is a unique factorization domain. The plant P is stabilizable if and only if the elementary factors of P are coprime, that is,*

$$\sum_{h \in H} (h) = \mathcal{A}. \quad \square$$

2.4. Causality of Given Plants. Here we define the causality of transfer functions, which is an important physical constraint, used in this paper. We employ the definition

of causality from Vidyasagar *et al.* [14, Definition 3.1] and introduce two terminologies used later frequently.

DEFINITION 2.6. *Let \mathcal{Z} be a prime ideal of \mathcal{A} , with $\mathcal{Z} \neq \mathcal{A}$, including all zerodivisors. Define the subsets \mathcal{P} and \mathcal{P}_S of \mathcal{F} as follows:*

$$\mathcal{P} = \{n/d \in \mathcal{F} \mid n \in \mathcal{A}, d \in \mathcal{A} \setminus \mathcal{Z}\}, \quad \mathcal{P}_S = \{n/d \in \mathcal{F} \mid n \in \mathcal{Z}, d \in \mathcal{A} \setminus \mathcal{Z}\}.$$

A transfer function in \mathcal{P} (\mathcal{P}_S) is called causal (strictly causal). Similarly, if every entry of a transfer matrix over \mathcal{F} is in \mathcal{P} (\mathcal{P}_S), the transfer matrix is called causal (strictly causal). A transfer matrix is said to be \mathcal{Z} -nonsingular if the determinant is in $\mathcal{A} \setminus \mathcal{Z}$, and \mathcal{Z} -singular otherwise.

In [14], the ideal \mathcal{Z} is not restricted to a prime ideal in general. On the other hand, in [11], the set of the denominators of causal transfer functions is a multiplicatively closed subset of \mathcal{A} . This property is natural since the multiplication of two causal transfer functions should be considered as causal one. Note that this multiplicativity is equivalent to \mathcal{Z} being prime provided that \mathcal{Z} is an ideal. By following the multiplicativity, we consider in this paper that \mathcal{Z} is prime.

In this paper, we do not assume that the prime spectrum of \mathcal{A} is irreducible and the plant P is strictly causal as in [11]. Alternatively, in the rest of the paper we assume only the following:

Assumption 2.7. The given plant is causal in the sense of Definition 2.6. \square

One can represent a causal plant P in the form of fractions $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$, where the matrices $N, D, \tilde{N}, \tilde{D}$ are over \mathcal{A} , and the matrices D, \tilde{D} are \mathcal{Z} -nonsingular.

3. Main Results. To state our results precisely we define the notion of generalized elementary factors, which is a generalization of the elementary factors in Definition 2.3. Then the main theorem will be presented.

Generalized Elementary Factors. Originally, the elementary factors have been defined over unique factorization domains as in Definition 2.3. The authors have enlarged this concept for integral domains[8] and have presented a criterion for feedback stabilizability over integral domains. We enlarge this concept again in the case of commutative rings.

Before stating the definition, we introduce several symbols used in the definition and widely in the rest of this paper. The symbol \mathcal{I} denotes the set of all sets of m distinct integers between 1 and $m + n$ (recall that m and n are the numbers of the inputs and the outputs, respectively). Normally, an element of \mathcal{I} will be denoted by I , possibly with suffixes such as integers. We will use an element of \mathcal{I} as a suffix as well as a set. For $I \in \mathcal{I}$, if i_1, \dots, i_m are elements of I in ascending order, that is, $i_a < i_b$ if $a < b$, then the symbol Δ_I denotes the matrix whose (k, i_k) -entry is 1 for $i_k \in I$ and zero otherwise.

DEFINITION 3.1. (Generalized Elementary Factors) *Let $P \in \mathcal{F}^{n \times m}$, and N and D are matrices over \mathcal{A} with $P = ND^{-1}$. Denote by T the matrix $\begin{bmatrix} N^t & D^t \end{bmatrix}^t$. For each $I \in \mathcal{I}$, define the ideal Λ_{PI} of \mathcal{A} by*

$$\Lambda_{PI} = \{\lambda \in \mathcal{A} \mid \exists K \in \mathcal{A}^{(m+n) \times m} \lambda T = K \Delta_I T\}.$$

We call the ideal Λ_{PI} the generalized elementary factor of the plant P with respect to I .

Whenever we use the symbol Λ with some suffix, it will denote a generalized elementary factor. We will also frequently use the symbols λ and λ_I with $I \in \mathcal{I}$ as particular elements of Λ_{PI} . Note that in Definitions 2.3 and 3.1, the matrices represented by T are different in general. However this difference is not a problem since the generalized elementary factors are independent of the choice of the fractions ND^{-1} as shown below.

LEMMA 3.2. *For any $I \in \mathcal{I}$, the generalized elementary factor of the plant P with respect to I is independent of the choice of matrices N and D over \mathcal{A} satisfying $P = ND^{-1}$.*

Proof. Let N, N', D be matrices over \mathcal{A} and d' be a scalar of \mathcal{A} such that $P = ND^{-1} =$

$N'd'^{-1}$ hold. Further, let

$$\begin{aligned}\Lambda_{PI1} &= \{\lambda \in \mathcal{A} \mid \exists K \in \mathcal{A}^{(m+n) \times m} \lambda \begin{bmatrix} N^t & D^t \end{bmatrix}^t = K \Delta_I \begin{bmatrix} N^t & D^t \end{bmatrix}^t\}, \\ \Lambda_{PI2} &= \{\lambda \in \mathcal{A} \mid \exists K \in \mathcal{A}^{(m+n) \times m} \lambda \begin{bmatrix} N'^t & d'E_m \end{bmatrix}^t = K \Delta_I \begin{bmatrix} N'^t & d'E_m \end{bmatrix}^t\}.\end{aligned}$$

In order to prove this lemma it is sufficient to show that the ideals Λ_{PI1} and Λ_{PI2} are equal. Suppose that λ is an element of Λ_{PI1} . Then there exists a matrix K such that $\lambda \begin{bmatrix} N^t & D^t \end{bmatrix}^t = K \Delta_I \begin{bmatrix} N^t & D^t \end{bmatrix}^t$. Multiplying $d'E_m$ on the right of both sides, we have $\lambda \begin{bmatrix} N'^t & d'E_m \end{bmatrix}^t D = K \Delta_I \begin{bmatrix} N'^t & d'E_m \end{bmatrix}^t D$. Since the matrix D is nonsingular, we have $\lambda \begin{bmatrix} N'^t & d'E_m \end{bmatrix}^t = K \Delta_I \begin{bmatrix} N'^t & d'E_m \end{bmatrix}^t$, so that $\lambda \in \Lambda_{PI2}$, which means that $\Lambda_{PI1} \subset \Lambda_{PI2}$. The opposite inclusion relation $\Lambda_{PI1} \supset \Lambda_{PI2}$ can be proved analogously. \square

Note also that for every I in \mathcal{I} , the generalized elementary factor of the plant with respect to I is not empty since it contains at least zero. In the case where the set of stable causal transfer functions is a unique factorization domain, the generalized elementary factor of the plant with the matrix $\Delta_I T$ being nonsingular becomes a principal ideal and the generator of its radical an elementary factor of the matrix T (in Definition 2.3) up to a unit multiple. Analogously, the elementary factor of the matrix W (in Definition 2.3) corresponds to the generalized elementary factor of the transposed plant P^t .

Main Results. We are now in position to state our main results.

THEOREM 3.3. *Consider a causal plant P . Then the following statements are equivalent:*

- (i) *The plant P is stabilizable.*
- (ii) *The \mathcal{A} -modules \mathcal{T}_P and \mathcal{W}_P are projective.*

(iii) *The set of all generalized elementary factors of P generates \mathcal{A} ; that is,*

$$(3.1) \quad \sum_{I \in \mathcal{I}} \Lambda_{PI} = \mathcal{A}.$$

□

In the theorem, (ii) and (iii) are criteria for feedback stabilizability. Comparing the theorem above with Theorems 2.4 and 2.5, we observe the following: (ii) and (iii) can be considered as generalizations of Theorems 2.4 and 2.5, respectively. For (ii), we do not assume as mentioned earlier that the prime spectrum of \mathcal{A} is irreducible and the plant P is strictly causal. The rank conditions of \mathcal{T}_P and \mathcal{W}_P are deleted. For (iii), the commutative ring \mathcal{A} is not restricted to a unique factorization domain. The elementary factors are replaced by the generalized elementary factors. Although two matrices T and W in Definition 2.3 are used to state Theorem 2.5, only one matrix T in Definition 3.1 is used in (iii).

We will present the proof of the theorem in § 5.

To make the notion of the generalized elementary factors familiar, we present here an example of the generalized elementary factors.

Example 3.4. On some synchronous high-speed electronic circuits such as computer memory devices, they cannot often have nonzero small delays (for example [5]). We suppose here that the system cannot have the unit delay as a nonzero small delay. Further we suppose that the impulse response of a transfer function being stable is finitely terminated. Thus the set \mathcal{A} becomes the set of polynomials generated by z^2 and z^3 , that is, $\mathcal{A} = \mathbb{R}[z^2, z^3]$, where z denotes the unit delay operator. Then \mathcal{A} is not a unique factorization domain but a Noetherian domain. The total field \mathcal{F} of fractions of \mathcal{A} is $\mathbb{R}(z^2, z^3)$, which is equal to $\mathbb{R}(z)$. The ideal \mathcal{Z} used to define the causality is given as the set of polynomials in $\mathbb{R}[z^2, z^3]$ whose constant terms are zero; that is, $\mathcal{Z} = z^2\mathcal{A} + z^3\mathcal{A} = \{az^2 + bz^3 \mid a, b \in \mathcal{A}\}$. Thus the set of causal transfer functions \mathcal{P} is given as n/d , where n, d are in \mathcal{A} and the constant term of d is

nonzero; that is, $\mathcal{P} = \{n/(a+bz^2+cz^3) \mid n \in \mathcal{A}, a \in \mathbb{R} \setminus \{0\}, b, c \in \mathcal{A}\}$. Further the set of strictly causal transfer functions \mathcal{P}_s is given as $\mathcal{P}_s = \{(a_1z^2+b_1z^3)/(a_2+b_2z^2+c_2z^3) \mid a_2 \in \mathbb{R} \setminus \{0\}, a_1, b_1, b_2, c_2 \in \mathcal{A}\}$

Since some factorized polynomials are sometimes expressed more compactly and easier to understand than the expanded ones, we here introduce the following notation: a polynomial in $\mathbb{R}[z]$ surrounded by “ \langle ” and “ \rangle ” indicates that it is in \mathcal{A} as well as in $\mathbb{R}[z]$ even though some factors between “ \langle ” and “ \rangle ” may not be in \mathcal{A} .

Let us consider the plant below:

$$(3.2) \quad P := \begin{bmatrix} (1-z^3)/(1-z^2) \\ (1-8z^3)/(1-4z^2) \end{bmatrix} \in \mathcal{P}^{2 \times 1}.$$

The representation of the plant is not unique. For example, the (1,1)-entry of the plant has an alternative form $(1+z^2+z^4)/(1+z^3)$ different from the expression in (3.2). Consider parameterizing the representation of the plant. To do so we consider the plant P over $\mathbb{R}(z)$ rather than over \mathcal{F} . Thus P can be expressed as

$$(3.3) \quad P = \begin{bmatrix} (1+z+z^2)/(1+z) \\ (1+2z+4z^2)/(1+2z) \end{bmatrix} \text{ over } \mathbb{R}(z).$$

However the coefficients of all numerators and denominators in (3.3) of z with degree 1 are not zero. To make them zero, we should multiply them by $(a_1(1-z) + b_1z^2 + c_1z^3)$ or $(a_2(1-2z) + b_2z^2 + c_2z^3)$ with $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathcal{A}$ as follows

$$(3.4) \quad P = \begin{bmatrix} \frac{\langle (1+z+z^2)(a_1(1-z)+b_1z^2+c_1z^3) \rangle}{\langle (1+z)(a_1(1-z)+b_1z^2+c_1z^3) \rangle} \\ \frac{\langle (1+2z+4z^2)(a_2(1-2z)+b_2z^2+c_2z^3) \rangle}{\langle (1+2z)(a_2(1-2z)+b_2z^2+c_2z^3) \rangle} \end{bmatrix}.$$

Every expression of the plant is given in the form of (3.4) with $a_1, b_1, c_1, a_2, b_2, c_2$ in \mathcal{A} provided that the denominators are not zero. From this, we have two observations. One is

that the plant P does not have its right- and left-coprime factorizations over \mathcal{A} (even so, it will be shown later that the plant is stabilizable). The other is that the elementary factor of this plant cannot be consistently defined over \mathcal{A} . Thus we employ the notion of the generalized elementary factor.

In the following, we calculate the generalized elementary factors of the plant. We choose the following matrices as N , D , and T used in Definition 3.1:

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} := N := \begin{bmatrix} (1 - z^3)(1 - 4z^2) \\ (1 - 8z^3)(1 - z^2) \end{bmatrix}, \\ [d] := D := [(1 - z^2)(1 - 4z^2)], \quad T := [N^t \quad D^t]^t.$$

Since $m = 1$ (the number of inputs) and $n = 2$ (the number of outputs), the set \mathcal{I} is given as $\mathcal{I} = \{\{1\}, \{2\}, \{3\}\}$ and we let $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3\}$.

Let us calculate the generalized elementary factor Λ_{PI_1} . Let $i_1 = 1$ so that $I_1 = \{i_1\}$. Then the $(1, i_1)$ -entry of the matrix Δ_{I_1} is 1 and the other entries are zero. Thus we have $\Delta_{I_1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. The generalized elementary factor Λ_{PI_1} is originally given as follows:

$$\begin{aligned} \Lambda_{PI_1} &= \{\lambda \in \mathcal{A} \mid \exists K \in \mathcal{A}^{(m+n) \times m} \lambda T = K \Delta_{I_1} T\} \\ (3.5) \quad &= \{\lambda \in \mathcal{A} \mid \exists k_1, k_2 \in \mathcal{A} \lambda \begin{bmatrix} n_2 & d \end{bmatrix}^t = n_1 \begin{bmatrix} k_1 & k_2 \end{bmatrix}^t\}. \end{aligned}$$

Consider (3.5) over $\mathbb{R}[z]$ instead of \mathcal{A} . Then the matrix equation in the set of (3.5) can be expressed as

$$\begin{aligned} \lambda \begin{bmatrix} (1 - z)(1 + z)(1 - 2z)(1 + 2z + 4z^2) \\ (1 - z)(1 + z)(1 - 2z)(1 + 2z) \end{bmatrix} &= \\ (3.6) \quad &(1 - z)(1 - 2z)(1 + 2z)(1 + z + z^2) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}. \end{aligned}$$

The set of λ 's such that there exist $k_1, k_2 \in \mathbb{R}[z]$ satisfying (3.6) is given as $\{(1+2z)(1+z+z^2)a \mid a \in \mathbb{R}[z]\}$, denoted by L_1 . Then the intersection of L_1 and \mathcal{A} is given as follows:

$$(3.7) \quad L_1 \cap \mathcal{A} = \{\langle (1+2z)(1+z+z^2)(a(1-3z) + bz^2 + cz^3) \rangle \in \mathcal{A} \mid a, b, c \in \mathcal{A}\}$$

This is equal to Λ_{PI_1} as shown below. First it is obvious that $L_1 \cap \mathcal{A} \supset \Lambda_{PI_1}$. For each $(1+2z)(1+z+z^2)(a(1-3z) + bz^2 + cz^3)$ with $a, b, c \in \mathcal{A}$, we have k_1 and k_2 as follows from (3.6):

$$\begin{aligned} k_1 &= (1+z)(1+2z+4z^2)(a(1-3z) + bz^2 + cz^3), \\ k_2 &= (1+z)(1+2z)(a(1-3z) + bz^2 + cz^3). \end{aligned}$$

Both k_1 and k_2 are in \mathcal{A} . Hence $L_1 \cap \mathcal{A} \subset \Lambda_{PI_1}$ and so $L_1 \cap \mathcal{A} = \Lambda_{PI_1}$. By (3.7), we can also consider that Λ_{PI_1} is generated by $\langle (1+2z)(1+z+z^2)(1-3z) \rangle$, $\langle (1+2z)(1+z+z^2)z^2 \rangle$, and $\langle (1+2z)(1+z+z^2)z^3 \rangle$.

Analogously, we can calculate the generalized elementary factors Λ_{PI_2} and Λ_{PI_3} of the plant with respect to I_2 and I_3 as follows:

$$\begin{aligned} \Lambda_{PI_2} &= \{\langle (1+z)(1+2z+4z^2)(a(1-3z) + bz^2 + cz^3) \rangle \in \mathcal{A} \mid a, b, c \in \mathcal{A}\}, \\ \Lambda_{PI_3} &= \{\langle (1+z)(1+2z)(a(1-3z) + bz^2 + cz^3) \rangle \in \mathcal{A} \mid a, b, c \in \mathcal{A}\}. \end{aligned}$$

Observe now that

$$\begin{aligned} \Lambda_{PI_1} &\ni \langle (1+2z)(1-3z)(1+z+z^2) \rangle =: \lambda_{0I_1}, \\ \Lambda_{PI_2} &\ni \langle (1+z)(1+2z+4z^2)(1-3z+z^2) \rangle =: \lambda_{0I_2} \end{aligned}$$

and further

$$\alpha_{I_1} \lambda_{0I_1} + \alpha_{I_2} \lambda_{0I_2} = 1,$$

where

$$\alpha_{I_1} = \frac{-4233-23646z^2-39836z^3-201780z^4-113016z^5+75344z^6}{5852} \in \mathcal{A},$$

$$\alpha_{I_2} = \frac{10085+18418z^2+121140z^3+131852z^4+113016z^5}{5852} \in \mathcal{A}.$$

Now let

$$(3.8) \quad \lambda_{I_1} := \alpha_{I_1} \lambda_{0I_1} \in \Lambda_{PI_1}, \quad \lambda_{I_2} := \alpha_{I_2} \lambda_{0I_2} \in \Lambda_{PI_2}.$$

Thus $\Lambda_{PI_1} + \Lambda_{PI_2} = \mathcal{A}$ and $\lambda_{I_1} + \lambda_{I_2} = 1$. Hence by Theorem 3.3, the plant P is stabilizable.

□

4. Intermediate Results. In this section we provide intermediate results which will be used in the proof of our main theorem stated in the preceding section. This section consists of three parts. We first show that a number of modules generated from plants, controllers, and feedback systems are isomorphic to one another. Next we develop the results which will help to show the existence of a well-defined stabilizing controller. We then give the coprime factorizability of the plant over \mathcal{A}_f , where f is an element of the generalized elementary factor of the plant.

Relationship in terms of Modules between Transfer Matrices P , C , and $H(P, C)$.

The first intermediate result is the relations, expressed in terms of modules, among the matrices P , C , and $H(P, C)$ as well as their transposed matrices. A number of modules are isomorphic to one another as follows.

PROPOSITION 4.1. *Suppose that P and C are matrices over $\mathcal{F}(\mathcal{R})$. Suppose further that $\det(E_n + PC)$ is a unit of $\mathcal{F}(\mathcal{R})$.*

(i) *The following \mathcal{R} -modules are isomorphic to one another:*

$$\mathcal{T}_{P,\mathcal{R}} \oplus \mathcal{T}_{C,\mathcal{R}}, \mathcal{T}_{H(P,C),\mathcal{R}}, \mathcal{T}_{H(P^t,C^t)^t,\mathcal{R}}, \mathcal{W}_{H(P^t,C^t),\mathcal{R}}, \mathcal{T}_{H(C,P),\mathcal{R}}.$$

(ii) *The following \mathcal{R} -modules are isomorphic to one another:*

$$\mathcal{W}_{P,\mathcal{R}} \oplus \mathcal{W}_{C,\mathcal{R}}, \mathcal{W}_{H(P,C),\mathcal{R}}, \mathcal{W}_{H(P^t,C^t)^t,\mathcal{R}}, \mathcal{T}_{H(P^t,C^t),\mathcal{R}}, \mathcal{W}_{H(C,P),\mathcal{R}}.$$

Further for a matrix X over $\mathcal{F}(\mathcal{R})$,

(iii) $\mathcal{T}_{X,\mathcal{R}} \simeq \mathcal{W}_{X^t,\mathcal{R}}$ and $\mathcal{W}_{X,\mathcal{R}} \simeq \mathcal{T}_{X^t,\mathcal{R}}$.

Note here that in the proposition above, the controller C need not be a stabilizing controller. For the case where C is a stabilizing controller, see Lemma 2 of [11].

We can consider that the proposition above, especially the relations $\mathcal{T}_{P,\mathcal{R}} \oplus \mathcal{T}_{C,\mathcal{R}} \simeq \mathcal{T}_{H(P,C),\mathcal{R}} \simeq \mathcal{T}_{H(C,P),\mathcal{R}}$, gives an interpretation of the structure of the feedback system in the sense that the module generated by the feedback system is given as the direct sum of the modules generated by the plant and the controller. In the proof (“(i)→(ii)”) of Theorem 3.3, this proposition will play a key role.

Proof. We first prove (iii). Let A and B be matrices over \mathcal{R} with $X = AB^{-1}$. Then we have $\mathcal{T}_{X,\mathcal{R}} \simeq M_r([A^t \ B^t]^t) \simeq M_c([A^t \ B^t]) \simeq \mathcal{W}_{(B^{-1})^t A^t, \mathcal{R}} \simeq \mathcal{W}_{X^t, \mathcal{R}}$. The other relation $\mathcal{W}_{X,\mathcal{R}} \simeq \mathcal{T}_{X^t, \mathcal{R}}$ can be proved in a similar way.

Next we prove (i). Suppose that $\det(E_n + PC)$ is a unit of $\mathcal{F}(\mathcal{R})$. We prove the following relations in order: (a) $\mathcal{T}_{P,\mathcal{R}} \oplus \mathcal{T}_{C,\mathcal{R}} \simeq \mathcal{T}_{H(P,C),\mathcal{R}}$, (b) $\mathcal{T}_{H(P,C),\mathcal{R}} \simeq \mathcal{T}_{H(P^t,C^t)^t,\mathcal{R}}$, (c) $\mathcal{T}_{H(P^t,C^t)^t,\mathcal{R}} \simeq \mathcal{W}_{H(P^t,C^t),\mathcal{R}}$, (d) $\mathcal{T}_{H(P,C),\mathcal{R}} \simeq \mathcal{T}_{H(C,P),\mathcal{R}}$.

(a) of (i). The proof of (a) follows mainly the proof of Lemma 2 in [11]. By virtue of Lemma 2.1, it is sufficient to show the relation $\mathcal{T}_{P,\mathcal{R}} \oplus \mathcal{T}_{C,\mathcal{R}} \simeq M_r([N_H^t \ d_H E_{m+n}]^t)$ with $N_H \in (\mathcal{R})_{m+n}$, $d_H \in \mathcal{R}$, where $H(P, C) = N_H d_H^{-1}$. Let N, N_C be matrices over \mathcal{R} and d ,

d_C be in \mathcal{R} with $P = Nd^{-1}$ and $C = N_C d_C^{-1}$. Further, let

$$Q = \begin{bmatrix} d_C E_n & N \\ -N_C & dE_m \end{bmatrix}, \quad S = \begin{bmatrix} d_C E_n & O \\ O & dE_m \end{bmatrix}.$$

From these we have $\mathcal{T}_{P,\mathcal{R}} \oplus \mathcal{T}_{C,\mathcal{R}} \simeq M_r([Q^t \ S^t]^t)$. In addition, since $\det(E_n + PC)$ is a unit of $\mathcal{F}(\mathcal{R})$, the matrix N_H is nonsingular, so that $M_r([Q^t \ S^t]^t) \simeq M_r([Q^t \ S^t]^t (\det(N_H)E_{m+n}))$ holds. A simple calculation shows that

$$\begin{bmatrix} Q \\ S \end{bmatrix} (\det(N_H)E_{m+n}) = \begin{bmatrix} d_H E_{m+n} \\ N_H \end{bmatrix} \text{adj}(N_H)S.$$

Because both matrices S and $\text{adj}(N_H)$ are nonsingular, we finally have that

$$\mathcal{T}_{P,\mathcal{R}} \oplus \mathcal{T}_{C,\mathcal{R}} \simeq M_r\left(\begin{bmatrix} Q \\ S \end{bmatrix}\right) \simeq M_r\left(\begin{bmatrix} Q \\ S \end{bmatrix} (\det(N_H)E_{m+n})\right) \simeq M_r\left(\begin{bmatrix} d_H E_{m+n} \\ N_H \end{bmatrix}\right) \simeq \mathcal{T}_{H(P,C),\mathcal{R}}.$$

(b) of (i). Observe that the following relation holds:

$$(4.1) \quad H(P^t, C^t)^t = \begin{bmatrix} O & E_m \\ E_n & O \end{bmatrix} H(P, C) \begin{bmatrix} O & E_n \\ E_m & O \end{bmatrix}.$$

Let N_H and d_H be a matrix over \mathcal{R} and a scalar of \mathcal{R} , respectively, with $H(P, C) = N_H d_H^{-1}$.

Then (4.1) can be rewritten as follows:

$$H(P^t, C^t)^t = \begin{bmatrix} O & E_m \\ E_n & O \end{bmatrix} N_H \left(\begin{bmatrix} O & E_m \\ E_n & O \end{bmatrix} d_H \right)^{-1}.$$

Hence, we have matrices A and B over \mathcal{R} with $H(P^t, C^t)^t = AB^{-1}$ such that

$$A = \begin{bmatrix} O & E_m \\ E_n & O \end{bmatrix} N_H, \quad B = \begin{bmatrix} O & E_m \\ E_n & O \end{bmatrix} d_H.$$

This gives the relation $\mathcal{T}_{H(P,C),\mathcal{R}} \simeq \mathcal{T}_{H(P^t,C^t)^t,\mathcal{R}}$.

(c) of (i). This is directly obtained by applying (iii) to the matrix $H(P^t, C^t)^t$.

(d) of (i). Between the matrices $H(P, C)$ and $H(C, P)$, the following relation holds:

$$H(C, P) = \begin{bmatrix} O & -E_n \\ E_m & O \end{bmatrix} H(P, C) \begin{bmatrix} O & E_m \\ -E_n & O \end{bmatrix}.$$

Letting N_H and d_H be a matrix over \mathcal{R} and a scalar of \mathcal{R} with $H(P, C) = N_H d_H^{-1}$ as in (b), we have matrices N'_H and D'_H over \mathcal{R} such that

$$\begin{bmatrix} N'_H \\ D'_H \end{bmatrix} = \begin{bmatrix} O & -E_n & & O \\ E_m & O & & \\ & & O & -E_n \\ & O & E_m & O \end{bmatrix} \begin{bmatrix} N_H \\ d_H E_{m+n} \end{bmatrix}$$

holds. Since $H(C, P) = N'_H D'^{-1}_H$ holds and the first matrix of the right-hand side of the equation above is invertible, the relation $\mathcal{T}_{H(P,C),\mathcal{R}} \simeq \mathcal{T}_{H(C,P),\mathcal{R}}$ holds.

Finally, arguments similar to (i) prove (ii). \square

Before moving to the next, we prove an easy lemma useful to give results for the transposed plants.

LEMMA 4.2. *A plant P is stabilizable if and only if its transposed plant P^t is. Moreover, in the case where the plant P is stabilizable, C is a stabilizing controller of P if and only if C^t is a stabilizing controller of the transposed plant P^t .*

Proof. (Only If) Suppose that a plant P is stabilizable. Let C be a stabilizing controller of P . First, (P^t, C^t) is in \widehat{F}_{ad} , since $(P, C) \in \widehat{F}_{\text{ad}}$ and $\det(E_n + PC) = \det(E_m + P^t C^t)$. From (4.1) in the proof of Proposition 4.1, if $H(P, C) \in (\mathcal{A})_{m+n}$, then $H(P^t, C^t) \in (\mathcal{A})_{m+n}$.

(If) Because $(P^t)^t = P$, the “If” part can be proved analogously. \square

\mathcal{Z} -nonsingularity of Transfer Matrices. In order to prove the stabilizability of the given causal plant, which will be necessary in the proof of the main theorem (Theorem 3.3), we should show the existence of the stabilizing controller. To do so, we will need to show that the denominator matrix of the stabilizing controller is nonsingular. The following result will help this matter.

LEMMA 4.3. *Suppose that there exist matrices A , B , C_1 , C_2 over \mathcal{A} such that the following square matrix is \mathcal{Z} -nonsingular:*

$$(4.2) \quad \begin{bmatrix} A & C_1 \\ B & C_2 \end{bmatrix},$$

where the matrix A is square and the matrices A and B have same number of columns. Then there exists a matrix R over \mathcal{A} such that the matrix $A + RB$ is \mathcal{Z} -nonsingular.

Before starting the proof, it is worth reviewing some easy facts about the prime ideal \mathcal{Z} .

Remark 4.4. (i) If a is in $\mathcal{A} \setminus \mathcal{Z}$ and expressed as $a = b + c$ with $b, c \in \mathcal{A}$, then at least one of b and c is in $\mathcal{A} \setminus \mathcal{Z}$. (ii) If a is in $\mathcal{A} \setminus \mathcal{Z}$ and b is in \mathcal{Z} , then the sum $a + b$ is in $\mathcal{A} \setminus \mathcal{Z}$. (iii) Every factor in \mathcal{A} of an element of $\mathcal{A} \setminus \mathcal{Z}$ belongs to $\mathcal{A} \setminus \mathcal{Z}$ (that is, if $a, b \in \mathcal{A}$ and $ab \in \mathcal{A} \setminus \mathcal{Z}$, then $a, b \in \mathcal{A} \setminus \mathcal{Z}$). \square

They will be used in the proofs of Lemma 4.3 and Theorem 3.3.

Proof of Lemma 4.3. This proof mainly follows that of Lemma 4.4.21 of [13].

If the matrix A itself is \mathcal{Z} -nonsingular, then we can select the zero matrix as R . Hence we assume in the following that A is \mathcal{Z} -singular.

Since (4.2) is \mathcal{Z} -nonsingular, there exists a full-size minor of $[A^t \ B^t]^t$ in $\mathcal{A} \setminus \mathcal{Z}$ by Laplace's expansion of (4.2) and Remark 4.4(i,iii). Let a be such a \mathcal{Z} -nonsingular full-size minor of $[A^t \ B^t]^t$ having as few rows from B as possible.

We here construct a matrix R such that $\det(A + RB) = \pm a + z$ with $z \in \mathcal{Z}$. Since A is \mathcal{Z} -singular, the full-size minor a must contain at least one row of B from the matrix $[A^t \ B^t]^t$.

Suppose that a is obtained by excluding the rows i_1, \dots, i_k of A and including the rows j_1, \dots, j_k of B , where both of i_1, \dots, i_k and j_1, \dots, j_k are in ascending order. Now define $R = (r_{ij})$ by $r_{i_1 j_1} = \dots = r_{i_k j_k} = 1$ and $r_{ij} = 0$ for all other i, j . Observe that $\det(A + RB)$ is expanded in terms of full-size minors of the matrices $[E \ R]$ and $[A^t \ B^t]^t$ from the factorization $A + RB = [E \ R][A^t \ B^t]^t$ by the Binet-Cauchy formula. Every minor of $[E \ R]$ containing more than k columns of R is zero. By the method of choosing the rows from $[A^t \ B^t]^t$ for the full-size minor a , every full-size minor of $[A^t \ B^t]^t$ having less than k rows of B is in \mathcal{Z} . There is only one nonzero minor of $[E \ R]$ containing exactly k columns of R , which is obtained by excluding the columns i_1, \dots, i_k of the identity matrix E and including the columns j_1, \dots, j_k of R ; it is equal to ± 1 . From the Binet-Cauchy formula, the corresponding minor of $[A^t \ B^t]^t$ is a . As a result, $\det(A + RB)$ is given as a sum of $\pm a$ and elements in \mathcal{Z} . By Remark 4.4(ii), the sum is in $\mathcal{A} \setminus \mathcal{Z}$ and so is $\det(A + RB)$. The matrix $A + RB$ is now \mathcal{Z} -nonsingular. \square

Coprimeness and Generalized Elementary Factors. We present here that for each non-nilpotent element λ of the generalized elementary factors, the plant has a right-coprime factorization over \mathcal{A}_λ . This will be independent of the stabilizability of the plant.

LEMMA 4.5. (cf. Proposition 2.2 of [9]) *Let Λ_{PI} be the generalized elementary factor of the plant P with respect to $I \in \mathcal{I}$ and further let $\sqrt{\Lambda_{PI}}$ denote the radical of Λ_{PI} (as an ideal). Suppose that λ is in $\sqrt{\Lambda_{PI}}$ but not nilpotent. Then, the \mathcal{A}_λ -module $\mathcal{T}_{P, \mathcal{A}_\lambda}$ is free of rank m .*

Proof. Let T, N, D be the matrices over \mathcal{A} as in Definition 3.1. Recall that $\mathcal{T}_{P, \mathcal{A}_\lambda}$ denotes the \mathcal{A}_λ -module generated by rows of the matrix T . By Definition 3.1, there exists a matrix K over \mathcal{A} such that $\lambda^r T = K \Delta_I T$ holds for some positive integer r . Then we have a factorization of the matrix T over \mathcal{A}_λ as $T = (\lambda^{-r} K)(\Delta_I T)$, where all entries of the matrix $\lambda^{-r} K$ are in \mathcal{A}_λ . In order to show that the \mathcal{A}_λ -module $\mathcal{T}_{P, \mathcal{A}_\lambda}$ is free of rank m , provided that λ is not

nilpotent, it is sufficient to prove the following two facts: (i) The matrix $\Delta_I T$ is nonsingular over \mathcal{A}_λ . (ii) There is a matrix X such that the matrix $[\lambda^{-r} K \quad X]$ is invertible over \mathcal{A}_λ and the matrix equation $T = [\lambda^{-r} K \quad X][(\Delta_I T)^t \quad O]^t$ holds.

(i). Observe that the matrix D is nonsingular over \mathcal{A}_λ as well as over \mathcal{A} . Since $D = \Delta_{\{n+1, \dots, m+n\}} T = (\lambda^{-r} \Delta_{\{n+1, \dots, m+n\}} K)(\Delta_I T)$ holds (note that the suffix of the symbol Δ is an ordered set of m distinct integers between 1 and $m+n$ as before Definition 3.1), the matrix $\Delta_I T$ is also nonsingular over \mathcal{A}_λ provided that λ is not nilpotent.

(ii). Let $\bar{i}_1, \dots, \bar{i}_n$ be n distinct integers in ascending order between 1 and $m+n$ excluding the integers in I . Then let X be the matrix whose (\bar{i}_k, k) -entry is 1 for each \bar{i}_k and zero otherwise. Then the determinant of $[\lambda^{-r} K \quad X]$ becomes ± 1 since the matrix $\lambda^{-r} \Delta_I K$ is the identity matrix of $(\mathcal{A}_\lambda)_m$. \square

The lemma below will be used in the proof (“(ii)→(iii)”) of the main theorem by letting $\mathcal{R} = \mathcal{A}_f$, where f is an element of the generalized elementary factor of the plant but not nilpotent.

LEMMA 4.6. *If \mathcal{R} -module $\mathcal{T}_{P, \mathcal{R}}(\mathcal{W}_{P, \mathcal{R}})$ is free of rank $m \binom{n}{n}$, there exist matrices A and B (\tilde{A} and \tilde{B}) over \mathcal{R} such that (A, B) is a right-coprime factorization (\tilde{A}, \tilde{B}) is a left-coprime factorization) of the plant $P \in \mathcal{F}(\mathcal{R})^{n \times m}$ over \mathcal{R} .*

Proof. This lemma is an analogy of the result given in the proof of Lemma 3 of [11]. See the proof of Lemma 3 of [11]. \square

Example 4.7. (Continued) Here we continue Example 3.4. Let us follow Lemmas 4.5 and 4.6 with the plant of (3.2). Let the notation be as in Example 3.4.

First we proceed along the proof of Lemma 4.5. As an example, we pick $I_1 \in \mathcal{I}$ as I and $\lambda_{I_1} \in \Lambda_{PI_1}$ as λ in the proof of Lemma 4.5. Recall that for each $\lambda \in \Lambda_{PI}$, there exists

a matrix K such that $\lambda T = K \Delta_I T$ holds. In the case of $\lambda_{I_1} \in \Lambda_{PI_1}$, the matrix K is given as

$$(4.3) \quad K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} \lambda_{I_1} \\ \alpha_{I_1} \langle (1+z)(1-3z)(1+2z+4z^2) \rangle \\ \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle \end{bmatrix}.$$

Thus we have the factorization $T = (\lambda^{-r} K)(\Delta_{I_1} T)$:

$$\begin{bmatrix} (1-z^3)(1-4z^2) \\ (1-8z^3)(1-z^2) \\ (1-z^2)(1-4z^2) \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1-3z)(1+2z+4z^2) \rangle \\ \lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle \end{bmatrix} [(1-z^3)(1-4z^2)],$$

where $r = 1$ and $\Delta_{I_1} T = [(1-z^3)(1-4z^2)]$. As shown in part (i) of the proof of Lemma 4.5, $\Delta_{I_1} T = [(1-z^3)(1-4z^2)]$ is nonsingular.

The matrix X in part (ii) of the proof of Lemma 4.5 is given as $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t$ by letting $\bar{i}_1 = 2$ and $\bar{i}_2 = 3$ according to $I_1 = \{1\}$. We can see that the matrix

$$(4.4) \quad [\lambda^{-1} K \quad X] = \begin{bmatrix} 1 & 0 & 0 \\ \lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1-3z)(1+2z+4z^2) \rangle & 1 & 0 \\ \lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle & 0 & 1 \end{bmatrix}$$

is invertible. Therefore the $\mathcal{A}_{\lambda_{I_1}}$ -module $\mathcal{T}_{P, \mathcal{A}_{\lambda_{I_1}}}$ is free and its rank is 1. (However we will see that the \mathcal{A} -module \mathcal{T}_P is not free, see Example 5.2)

From (4.4) and the matrix equation $T = [\lambda^{-r} K \quad X][(\Delta_{I_1} T)^t \quad O]^t$, we let

$$(4.5) \quad \begin{bmatrix} N_{I_1} \\ D_{I_1} \end{bmatrix} := \begin{bmatrix} 1 \\ \lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1-3z)(1+2z+4z^2) \rangle \\ \lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle \end{bmatrix} = \lambda_{I_1}^{-1} K,$$

$$(4.6) \quad \begin{bmatrix} \tilde{Y}_{I_1} & \tilde{X}_{I_1} \\ \times & \times \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 \\ -\lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1-3z)(1+2z+4z^2) \rangle & 1 & 0 \\ -\lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle & 0 & 1 \end{bmatrix} \\ = [\lambda_{I_1}^{-1} K \quad X]^{-1},$$

where $N_{I_1} \in \mathcal{A}_{\lambda_{I_1}}^{2 \times 1}$, $\tilde{Y}_{I_1} \in \mathcal{A}_{\lambda_{I_1}}^{1 \times 2}$, $D_{I_1}, \tilde{X}_{I_1} \in (\mathcal{A}_{\lambda_{I_1}})_1$, and \times denotes some matrix. Then (N_{I_1}, D_{I_1}) is a right-coprime factorization of the plant over $\mathcal{A}_{\lambda_{I_1}}$ with $\tilde{Y}_{I_1} N_{I_1} + \tilde{X}_{I_1} D_{I_1} = E_1$, which is consistent with Lemma 4.6. \square

5. Proof of Main Results.

Now we give the proof of the main theorem.

Proof of Theorem 3.3. We prove the following relations in order: (a) “(i)→(ii),” (b) “(ii)→(iii),” and (c) “(iii)→(i).”

(a) “(i)→(ii)”: Suppose that C is a stabilizing controller of the plant P . Then, the \mathcal{A} -module $\mathcal{T}_{H(P,C)}$ is obviously free. By the relation $\mathcal{T}_{P,\mathcal{R}} \oplus \mathcal{T}_{C,\mathcal{R}} \simeq \mathcal{T}_{H(P,C),\mathcal{R}}$ in Proposition 4.1(i), we have that the \mathcal{A} -module \mathcal{T}_P is projective. By using Proposition 4.1(iii) and Lemma 4.2, the projectivity of the \mathcal{A} -module \mathcal{W}_P can be proved analogously.

(b) “(ii)→(iii)”: Suppose that (ii) holds, that is, the modules \mathcal{T}_P and \mathcal{W}_P are projective. We let T, N, D be the matrices over \mathcal{A} as in Definition 3.1. According to Theorem IV.32 of [6, p.295], there exist finite sets F_1 and F_2 such that (1) each of them generates \mathcal{A} and (2) for any $f \in F_1$ ($f \in F_2$), the \mathcal{A}_f -module $\mathcal{T}_{P,\mathcal{A}_f}$ ($\mathcal{W}_{P,\mathcal{A}_f}$) is free. Let F be the set of all $f_1 f_2$ ’s with $f_1 \in F_1$ and $f_2 \in F_2$. Then F generates \mathcal{A} again, and the \mathcal{A}_f -modules $\mathcal{T}_{P,\mathcal{A}_f}$ and $\mathcal{W}_{P,\mathcal{A}_f}$ are free for every $f \in F$. We suppose without loss of generality that the sets F_1, F_2 , and F do not contain any nilpotent element because $1+x$ is a unit of \mathcal{A} for any nilpotent x (cf. [2, p.10]). (However, we note that other zerodivisors cannot be excluded from the set F .) The rank of the free \mathcal{A}_f -module $\mathcal{T}_{P,\mathcal{A}_f}$ is m , since m rows of the denominator matrix D are independent over \mathcal{A}_f as well as over \mathcal{A} . Analogously the rank of $\mathcal{W}_{P,\mathcal{A}_f}$ is n .

In order to show that (iii) holds, it suffices to show that the relation $\sum_{f \in F} (f^\xi) \subset$

$\sum_{I \in \mathcal{I}} \Lambda_{PI}$ holds for a sufficiently large integer ξ . Once this relation is obtained, since $\sum_{f \in F} (f^\xi) = \mathcal{A}$ holds, we have $\sum_{I \in \mathcal{I}} \Lambda_{PI} = \mathcal{A}$.

Let f be an arbitrary but fixed element of F . Let V_f be a square matrix of size m whose rows are m distinct generators of the \mathcal{A}_f -module $M_r([N^t \ D^t]^t) (\simeq \mathcal{T}_{P, \mathcal{A}_f})$. We assume without loss of generality that V_f is over \mathcal{A} , that is, the denominators of all entries of V_f are 1. Otherwise if V_f is over \mathcal{A}_f but not over \mathcal{A} , V_f multiplied by f^x , with a sufficiently large integer x , will be over \mathcal{A} , so that we can consider such $V_f f^x$ as “ V_f .” Thus the following matrix equation holds over \mathcal{A} :

$$(5.1) \quad f^\nu T = K_f V_f$$

with a nonnegative integer ν and a matrix $K_f \in \mathcal{A}^{(m+n) \times m}$.

In order to prove the relation $\sum_{f \in F} (f^\xi) \subset \sum_{I \in \mathcal{I}} \Lambda_{PI}$, we will first show the relation

$$(5.2) \quad I_{m\mathcal{A}}(f^\nu K_f) \subset \sum_{I \in \mathcal{I}} \Lambda_{PI}$$

and then

$$(5.3) \quad (f^\xi) \subset I_{m\mathcal{A}}(f^\nu K_f).$$

Observe first that $\det(f^\nu \Delta_I K_f) \in \Lambda_{PI}$ because

$$\det(f^\nu \Delta_I K_f) T = f^{m\nu} K_f \operatorname{adj}(\Delta_I K_f) \Delta_I T.$$

Since every element of $I_{m\mathcal{A}}(f^\nu K_f)$ is an \mathcal{A} -linear combination of $\det(f^\nu \Delta_I K_f)$'s for all $I \in \mathcal{I}$, we have (5.2).

We next present (5.3). Let N_0 and D_0 be matrices with $K_f = [N_0^t \ D_0^t]^t$. Since each row of V_f is generated by rows of $[N^t \ D^t]^t$ over \mathcal{A}_f , there exist matrices \tilde{Y}_0 and \tilde{X}_0 over \mathcal{A}_f such that $V_f = [\tilde{Y}_0 f^\nu \ \tilde{X}_0 f^\nu] [N^t \ D^t]^t$. Thus, since V_f is nonsingular over \mathcal{A}_f , we have

$[\tilde{Y}_0 \ \tilde{X}_0][N_0^t \ D_0^t]^t = E_m$, which implies that (N_0, D_0) is a right-coprime factorization of the plant P over \mathcal{A}_f . Recall here that $\mathcal{W}_{P, \mathcal{A}_f}$ is free of rank n . Thus by Lemma 4.6 there exist matrices \tilde{N}_0 and \tilde{D}_0 such that $(\tilde{N}_0, \tilde{D}_0)$ is a left-coprime factorization of the plant P over \mathcal{A}_f . Let Y_0 and X_0 be matrices over \mathcal{A}_f such that $\tilde{N}_0 Y_0 + \tilde{D}_0 X_0 = E_n$ holds. Then we have the following matrix equation:

$$(5.4) \quad \begin{bmatrix} \tilde{Y}_0 & \tilde{X}_0 \\ -\tilde{D}_0 & \tilde{N}_0 \end{bmatrix} \begin{bmatrix} N_0 & -X_0 \\ D_0 & Y_0 \end{bmatrix} = \begin{bmatrix} E_m & -\tilde{Y}_0 X_0 + \tilde{X}_0 Y_0 \\ O & E_n \end{bmatrix}.$$

Denote by R the matrix $[-X_0^t \ Y_0^t]^t$. Then the matrix $[K_f \ R]$ is invertible over \mathcal{A}_f since the right-hand side of (5.4) is invertible. For each $I \in \mathcal{I}$, let \bar{I} be the ordered set of n distinct integers between 1 and $m+n$ excluding m integers in I and let $\bar{i}_1, \dots, \bar{i}_n$ be elements of \bar{I} in ascending order. Let $\Delta_{\bar{I}} \in \mathcal{A}^{m \times (m+n)}$ denote the matrix whose (k, \bar{i}_k) -entry is 1 for $\bar{i}_k \in \bar{I}$ and zero otherwise. Then, by using Laplace's expansion, the following holds:

$$\det([K_f \ R]) = \sum_{I \in \mathcal{I}} (\pm \det(\Delta_I K_f) \det(\Delta_{\bar{I}} R)),$$

which is a unit of \mathcal{A}_f . From this and since the ideal $I_{m\mathcal{A}_f}(K_f)$ is generated by $\det(\Delta_I K_f)$'s for all $I \in \mathcal{I}$, we have $I_{m\mathcal{A}_f}(K_f) = \mathcal{A}_f$. This equality over \mathcal{A}_f gives (5.3) for a sufficiently large integer ξ .

From (5.2) and (5.3), the relation $\sum_{f \in F} (f^\xi) \subset \sum_{I \in \mathcal{I}} \Lambda_{PI}$ holds. Therefore we conclude that the relation $\sum_{I \in \mathcal{I}} \Lambda_{PI} = \mathcal{A}$ holds.

(c) “(iii)→(i)”: To prove the stabilizability, we will construct a stabilizing controller of the causal plant P from right-coprime factorizations over \mathcal{A}_f with some f 's in \mathcal{A} . Let N and D be matrices over \mathcal{A} such that $P = ND^{-1}$ and D is \mathcal{Z} -nonsingular. From (3.1), there exist λ_I 's such that $\sum \lambda_I = 1$, where λ_I is an element of generalized elementary factor Λ_{PI} of the plant P with respect to I in \mathcal{I} ; that is, $\lambda_I \in \Lambda_{PI}$. Now let these λ_I 's be fixed. Further,

let $\mathcal{I}^\#$ be the set of I 's of these nonzero λ_I 's; that is, $\sum_{I \in \mathcal{I}^\#} \lambda_I = 1$. As in (b), we can consider without loss of generality that for every $I \in \mathcal{I}^\#$, λ_I is not a nilpotent element of \mathcal{A} . For each $I \in \mathcal{I}^\#$, the \mathcal{A}_{λ_I} -module $\mathcal{T}_{P, \mathcal{A}_{\lambda_I}}$ is free of rank m by Lemma 4.5. As in (b) again, let V_{λ_I} be a square matrix of size m whose rows are m distinct generators of the \mathcal{A}_{λ_I} -module $M_r([N^t \ D^t]^t) (\simeq \mathcal{T}_{P, \mathcal{A}_{\lambda_I}})$ and assume without loss of generality that V_{λ_I} is over \mathcal{A} . Then there exist matrices $\tilde{X}_I, \tilde{Y}_I, N_I, D_I$ over \mathcal{A}_{λ_I} such that

$$(5.5) \quad \begin{aligned} [N^t \ D^t]^t &= [N_I^t \ D_I^t]^t V_{\lambda_I}, \quad [\tilde{Y}_I \ \tilde{X}_I] [N^t \ D^t]^t = V_{\lambda_I}, \\ \tilde{Y}_I N_I + \tilde{X}_I D_I &= E_m \end{aligned}$$

with $P = N_I D_I^{-1}$ over $\mathcal{F}(\mathcal{A}_{\lambda_I})$.

We here present a formula of a stabilizing controller which is constructed from the matrices \tilde{X}_I and \tilde{Y}_I . For any positive integer ω , there are coefficients a_I 's in \mathcal{A} with $\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega = 1$. Let ω be a sufficiently large integer. Thus the matrices $\lambda_I^\omega D_I \tilde{X}_I$ and $\lambda_I^\omega D_I \tilde{Y}_I$ are over \mathcal{A} for every $I \in \mathcal{I}^\#$. The stabilizing controller we will construct has the form

$$(5.6) \quad C = \left(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \tilde{X}_I \right)^{-1} \left(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \tilde{Y}_I \right).$$

In the following we first consider that the matrix $(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \tilde{X}_I)$ is \mathcal{Z} -nonsingular and show that the plant is stabilized by the matrix C of (5.6). After showing it, we will be concerned with the case where the matrix $(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \tilde{X}_I)$ is \mathcal{Z} -singular.

Suppose that the matrix $(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \tilde{X}_I)$ is \mathcal{Z} -nonsingular. To prove that C is a stabilizing controller of P , it is sufficient to show that $(P, C) \in \hat{F}_{\text{ad}}$ and that four blocks of (2.1) are over \mathcal{A} .

We first show that $(P, C) \in \widehat{F}_{\text{ad}}$. The following matrix equation holds:

$$\begin{aligned} E_m + CP &= E_m + \left(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \widetilde{X}_I \right)^{-1} \left(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \widetilde{Y}_I \right) N D^{-1} \\ &= \left(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \widetilde{X}_I \right)^{-1} \\ &\quad \left(\left(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \widetilde{X}_I \right) D + \left(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \widetilde{Y}_I \right) N \right) D^{-1}. \end{aligned}$$

By the (1,1)-block of (5.8), we have

$$(5.7) \quad E_m + CP = \left(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \widetilde{X}_I \right)^{-1}.$$

This shows that $\det(E_m + CP)$ is a unit of \mathcal{F} so that $(P, C) \in \widehat{F}_{\text{ad}}$.

Next we show that four blocks of (2.1) are over \mathcal{A} . The (2,2)-block is the inverse of (5.7):

$$(E_m + CP)^{-1} = \sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \widetilde{X}_I.$$

Similarly, simple calculations show that other blocks are also over \mathcal{A} as follows:

(2,1)-block:

$$C(E_n + PC)^{-1} = \sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \widetilde{Y}_I,$$

(1,1)-block:

$$(E_n + PC)^{-1} = E_n - \sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega N_I \widetilde{Y}_I,$$

(1,2)-block:

$$-P(E_m + CP)^{-1} = - \sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega N_I \widetilde{X}_I.$$

To finish the proof, we proceed to deal with the case where the matrix $(\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \widetilde{X}_I)$ is \mathcal{Z} -singular. To make the matrix \mathcal{Z} -nonsingular, we reconstruct the matrices \widetilde{X}_I and \widetilde{Y}_I with

an $I \in \mathcal{I}^\sharp$.

Since the sum of $a_I \lambda_I^\omega$'s for $I \in \mathcal{I}^\sharp$ is equal to 1, by Remark 4.4(i, iii) there exists at least one summand, say $a_{I_0} \lambda_{I_0}^\omega$ with an $I_0 \in \mathcal{I}^\sharp$, such that both a_{I_0} and λ_{I_0} belong to $\mathcal{A} \setminus \mathcal{Z}$. Let R_{I_0} be a parameter matrix of $\mathcal{A}_{\lambda_{I_0}}^{m \times n}$. Then the following matrix equation holds over $\mathcal{A}_{\lambda_{I_0}}$:

$$(\tilde{Y}_{I_0} + R_{I_0} \tilde{D}_{I_0}) N_{I_0} + (\tilde{X}_{I_0} - R_{I_0} \tilde{N}_{I_0}) D_{I_0} = E_m,$$

where $\tilde{D}_{I_0} = \det(\lambda_{I_0}^\omega D_{I_0}) E_n$ and $\tilde{N}_{I_0} = \lambda_{I_0}^\omega N_{I_0} \operatorname{adj}(\lambda_{I_0}^\omega D_{I_0})$. Since ω is a sufficiently large integer, the following matrix equation is over \mathcal{A} :

$$\begin{aligned} & \left(\lambda_{I_0}^\omega (\tilde{Y}_{I_0} + R_{I_0} \tilde{D}_{I_0}) \right) \left(\lambda_{I_0}^\omega N_{I_0} \right) \\ & + \left(\lambda_{I_0}^\omega (\tilde{X}_{I_0} - R_{I_0} \tilde{N}_{I_0}) \right) \left(\lambda_{I_0}^\omega D_{I_0} \right) = \lambda_{I_0}^{2\omega} E_m, \end{aligned}$$

where the matrices surrounded by “(” and “)” in the left-hand side are over \mathcal{A} . From the first matrix equation of (5.5), $\det(D) = \det(D_I) \det(V_{\lambda_I})$ over \mathcal{A}_{λ_I} for every $I \in \mathcal{I}^\sharp$. Thus by Remark 4.4(iii) the matrix $\lambda_{I_0}^\omega D_{I_0}$ is \mathcal{Z} -nonsingular and so is the matrix $\tilde{D}_{I_0} (= \det(\lambda_{I_0}^\omega D_{I_0}) E_n)$.

Consider now the following matrix equation over \mathcal{A} :

$$(5.8) \quad \begin{bmatrix} \sum_{I \in \mathcal{I}^\sharp} a_I \lambda_I^\omega D_I \tilde{X}_I & \sum_{I \in \mathcal{I}^\sharp} a_I \lambda_I^\omega D_I \tilde{Y}_I \\ -a_{I_0} \lambda_{I_0}^\omega \det(\lambda_{I_0}^\omega D_{I_0}) \tilde{N}_{I_0} & a_{I_0} \lambda_{I_0}^\omega \det(\lambda_{I_0}^\omega D_{I_0}) \tilde{D}_{I_0} \end{bmatrix} \begin{bmatrix} D & O \\ N & E_n \end{bmatrix} \\ = \begin{bmatrix} D & \sum_{I \in \mathcal{I}^\sharp} a_I \lambda_I^\omega D_I \tilde{Y}_I \\ O & a_{I_0} \lambda_{I_0}^\omega \det(\lambda_{I_0}^\omega D_{I_0}) \tilde{D}_{I_0} \end{bmatrix}.$$

The $(1, 1)$ -block of (5.8) can be understood in the following way. From the last matrix equation in (5.5) we have the following matrix equation over \mathcal{A}_{λ_I} :

$$D_I \tilde{Y}_I N + D_I \tilde{X}_I D = D.$$

Considering the above equation multiplied by $a_I \lambda_I^\omega$ over \mathcal{A} , we have the following equation over \mathcal{A} :

$$(5.9) \quad a_I \lambda_I^\omega D_I \tilde{Y}_I N + a_I \lambda_I^\omega D_I \tilde{X}_I D = a_I \lambda_I^\omega D + a_I \lambda_I^\omega Z,$$

where Z is a matrix over \mathcal{A} such that $\lambda_I^x Z$ is the zero matrix for some positive integer x . Since ω is a large positive integer, we can consider that the matrix $a_I \lambda_I^\omega Z$ in (5.9) becomes the zero matrix. Therefore, the $(1, 1)$ -block of (5.8) holds. Then the matrix of the right-hand side of (5.8) is \mathcal{Z} -nonsingular since both of the matrices D and $a_{I_0} \lambda_{I_0}^\omega \det(\lambda_{I_0}^\omega D_{I_0}) \tilde{D}_{I_0}$ in the right-hand side of (5.8) are \mathcal{Z} -nonsingular. Hence the first matrix of (5.8) is also \mathcal{Z} -nonsingular by Remark 4.4(iii). By Lemma 4.3 and (5.8), there exists a matrix R'_{I_0} of $\mathcal{A}^{m \times n}$ such that the following matrix is \mathcal{Z} -nonsingular:

$$(5.10) \quad \sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \tilde{X}_I - a_{I_0} \lambda_{I_0}^{2\omega} D_{I_0} \operatorname{adj}(\lambda_{I_0}^\omega D_{I_0}) R'_{I_0} \tilde{N}_{I_0}.$$

Let now $R_{I_0} := \lambda_{I_0}^\omega \operatorname{adj}(\lambda_{I_0}^\omega D_{I_0}) R'_{I_0}$, $\tilde{X}_{I_0} := \tilde{X}_{I_0} - R_{I_0} \tilde{N}_{I_0}$, and $\tilde{Y}_{I_0} := \tilde{Y}_{I_0} + R_{I_0} \tilde{D}_{I_0}$. Then the matrix $\sum_{I \in \mathcal{I}^\#} a_I \lambda_I^\omega D_I \tilde{X}_I$ becomes equal to (5.10) and \mathcal{Z} -nonsingular \square

Remark 5.1. From the proof above, if (i) we can check (3.1) and if (ii) we can construct the right-coprime factorizations of the given causal plant over \mathcal{A}_{λ_I} for every $I \in \mathcal{I}^\#$, then we can construct stabilizing controllers of the plant, where λ_I is an element of the generalized elementary factor of the plant. For (i), if we can compute, for example, the Gröbner basis[4] over \mathcal{A} and if the generalized elementary factors of the plant are finitely generated, (3.1) can be checked. For (ii), it is already known by Lemmas 4.6 and 4.5 that there exist the right-coprime factorizations of the plant over \mathcal{A}_{λ_I} .

Let us give an example concerning the Gröbner basis. Consider the generalized elemen-

tary factors of Example 3.4. They are expressed as

$$\begin{aligned}
\Lambda_{PI_1} &= (\langle (1+2z)(1+z+z^2)(1-3z) \rangle) + (\langle (1+2z)(1+z+z^2)z^2 \rangle) \\
&\quad + (\langle (1+2z)(1+z+z^2)z^3 \rangle), \\
\Lambda_{PI_2} &= (\langle (1+z)(1+2z+4z^2)(1-3z) \rangle) + (\langle (1+z)(1+2z+4z^2)z^2 \rangle) \\
&\quad + (\langle (1+z)(1+2z+4z^2)z^3 \rangle), \\
\Lambda_{PI_3} &= (\langle (1+z)(1+2z)(1-3z) \rangle) + (\langle (1+z)(1+2z)z^2 \rangle) \\
&\quad + (\langle (1+z)(1+2z)z^3 \rangle).
\end{aligned}$$

Hence each of them has three generators and so is finitely generated. Suppose here that we can calculate the Gröbner basis over \mathcal{A} (of Example 3.4). Then as above the plant is stabilizable if and only if the Gröbner basis of the nine generators contains 1. \square

In the following two examples we follow the proof of Theorem 3.3. In the first one, we construct a stabilizing controller with part (c). In the other example, we follow part (b). On the other hand we do not follow part (a) since it can be followed easily with part (a) of (i) in the proof of Proposition 4.1.

Example 5.2. (Continued) We continue Example 3.4 (and 4.7) and construct a stabilizing controller of the plant as in “(iii)→(i)” of the proof above. Let the notation be as in Examples 3.4 and 4.7.

Since, in this example, $\Lambda_{PI_1} + \Lambda_{PI_2} = \mathcal{A}$ holds, $\mathcal{I}^\# = \{I_1, I_2\}$. For $I_1 \in \mathcal{I}^\#$, the matrices N_{I_1} , D_{I_1} , \tilde{X}_{I_1} and \tilde{Y}_{I_1} of (5.5) over $\mathcal{A}_{\lambda_{I_1}}$ have been calculated as (4.5) and (4.6). For $I_2 \in \mathcal{I}^\#$, the matrices N_{I_2} , D_{I_2} , \tilde{X}_{I_2} and \tilde{Y}_{I_2} of (5.5) over $\mathcal{A}_{\lambda_{I_2}}$ can be calculated analogously as follows:

$$N_{I_2} = \begin{bmatrix} \lambda_{I_2}^{-1} \alpha_{I_2} \langle (1+2z)(1-3z+z^2)(1+z+z^2) \rangle \\ 1 \end{bmatrix},$$

$$D_{I_2} = [\lambda_{I_2}^{-1} \alpha_{I_2} \langle (1+z)(1+2z)(1-3z+z^2) \rangle],$$

$$\tilde{Y}_{I_2} = [0 \quad 1], \tilde{X}_{I_2} = [0].$$

Then the following matrices are over \mathcal{A} :

$$\lambda_{I_1} D_{I_1} \tilde{X}_{I_1} = [0], \quad \lambda_{I_1} D_{I_1} \tilde{Y}_{I_1} = [\alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle \quad 0],$$

$$\lambda_{I_2} D_{I_2} \tilde{X}_{I_2} = [0], \quad \lambda_{I_2} D_{I_2} \tilde{Y}_{I_2} = [0 \quad \alpha_{I_2} \langle (1+z)(1+2z)(1-3z+z^2) \rangle].$$

Hence in this example, we can let $\omega = 1$ as a sufficiently large integer and $a_I = 1$ for all $I \in \mathcal{I}^\sharp$ (since $\sum_{I \in \mathcal{I}^\sharp} \lambda_I^\omega = 1$).

Note here that the matrix $\lambda_{I_1} D_{I_1} \tilde{X}_{I_1} + \lambda_{I_2} D_{I_2} \tilde{X}_{I_2}$ is \mathcal{Z} -singular. Hence we should reconstruct the matrices \tilde{Y}_{I_i} and \tilde{X}_{I_i} with i being either 1 or 2 as in the proof of Theorem 3.3. Since, in this example, both λ_{I_1} and λ_{I_2} are nonzerodivisors, we can choose each of 1 and 2. This example proceeds by reconstructing the matrices \tilde{Y}_{I_1} and \tilde{X}_{I_1} , which means that I_1 is used as I_0 in the proof of Theorem 3.3. The actual reconstruction is done by following the proof of Lemma 4.3.

Consider the first matrix of (5.8). Recall that $\tilde{N}_{I_0} = \lambda_{I_0}^\omega N_{I_0} \text{adj}(\lambda_{I_0}^\omega D_{I_0})$ and $\tilde{D}_{I_0} = \det(\lambda_{I_0}^\omega D_{I_0}) E_n$. In this example, they are given as

$$\tilde{N}_{I_1} = (\tilde{N}_{I_0}) \begin{bmatrix} \lambda_{I_1} \\ \alpha_{I_1} \langle (1+z)(1-3z)(1+2z+4z^2) \rangle \end{bmatrix},$$

$$\tilde{D}_{I_1} = (\tilde{D}_{I_0}) \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle E_2.$$

One can check that the first matrix of (5.8) is \mathcal{Z} -nonsingular. Then we construct a matrix R'_{I_0} of $\mathcal{A}^{1 \times 2}$ such that (5.10) is \mathcal{Z} -nonsingular. To do so, we follow temporarily the proof of the Lemma 4.3.

Consider the first matrix of (5.8) as the matrix of (4.2), that is,

$$\begin{aligned}
 A &= \sum_{I \in \mathcal{I}^\sharp} a_I \lambda_I^\omega D_I \tilde{X}_I = [0], \\
 B &= -a_{I_0} \lambda_{I_0}^\omega \det(\lambda_{I_0}^\omega D_{I_0}) \tilde{N}_{I_0} \\
 &= -\alpha_{I_1} \lambda_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle \begin{bmatrix} \lambda_{I_1} \\ \alpha_{I_1} \langle (1+z)(1-3z)(1+2z+4z^2) \rangle \end{bmatrix}.
 \end{aligned}$$

Then we choose a full-size a minor of $[A^t \ B^t]^t$ having as few rows from B as possible. In this example, we can choose both entries in B . Here we choose the $(1, 1)$ -entry of B , so that

$$(5.11) \quad a = -\alpha_{I_1} \lambda_{I_1}^2 \langle (1+z)(1+2z)(1-3z) \rangle.$$

Thus we have $k = 1$, $i_1 = 1$ and $j_1 = 1$, where the notations k , i_1, \dots, i_k , and j_1, \dots, j_k are as in the proof of Lemma 4.3. Hence R in the proof is given as $R = \begin{bmatrix} 1 & 0 \end{bmatrix}$. We can confirm that $A + RB = \begin{bmatrix} a \end{bmatrix}$ which is \mathcal{Z} -nonsingular by observing that every factor of the right-hand side of (5.11) has a nonzero constant term.

From here on we proceed with following again the proof of Theorem 3.3. The notation R used above corresponds to the notation R'_{I_0} in the proof of Theorem 3.3 (that is, $R'_{I_0} = \begin{bmatrix} 1 & 0 \end{bmatrix}$). The matrix R_{I_1} is given as follows:

$$R_{I_1} = (R_{I_0} =) \lambda_{I_1}^\omega \operatorname{adj}(\lambda_{I_1}^\omega D_{I_1}) R'_{I_1} = \lambda_{I_1} \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Then new \tilde{X}_{I_1} and \tilde{Y}_{I_1} are given as follows:

$$\begin{aligned}
 \tilde{X}_{I_1} &:= \tilde{X}_{I_1} - R_{I_1} \tilde{N}_{I_1} = \begin{bmatrix} -\lambda_{I_1}^2 \end{bmatrix}, \\
 \tilde{Y}_{I_1} &:= \tilde{Y}_{I_1} + R_{I_1} \tilde{D}_{I_1} = \begin{bmatrix} 1 + \alpha_{I_1} \lambda_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle & 0 \end{bmatrix}.
 \end{aligned}$$

Therefore a stabilizing controller C of the form (5.6) is obtained as

$$C = (\lambda_{I_1} D_{I_1} \tilde{X}_{I_1} + \lambda_{I_2} D_{I_2} \tilde{X}_{I_2})^{-1} (\lambda_{I_1} D_{I_1} \tilde{Y}_{I_1} + \lambda_{I_2} D_{I_2} \tilde{Y}_{I_2})$$

$$= \frac{-1}{\alpha_{I_1} \lambda_{I_1}^2 \langle (1+z)(1+2z)(1-3z) \rangle}$$

$$\begin{bmatrix} \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle (1 + \alpha_{I_1} \lambda_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle) \\ \alpha_{I_2} \langle (1+z)(1+2z)(1-3z+z^2) \rangle \end{bmatrix}^t.$$

The matrix $H(P, C)$ with the stabilizing controller C above over \mathcal{A} is expressed as follows:

$$H(P, C) = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix},$$

where

$$h_{11} = -\alpha_{I_1} \lambda_{I_1}^2 \langle (1+z)(1+2z)(1-3z) \rangle$$

$$+ \alpha_{I_2} \langle (1+z)(1-3z+z^2)(1+2z+4z^2) \rangle,$$

$$h_{12} = -\alpha_{I_2} \langle (1+2z)(1+z+z^2)(1-3z+z^2) \rangle,$$

$$h_{13} = \lambda_{I_1}^3,$$

$$h_{21} = -\alpha_{I_1} \langle (1+z)(1-3z)(1+2z+4z^2) \rangle$$

$$(1 + \lambda_{I_1} \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle),$$

$$h_{22} = \alpha_{I_1} (\langle (1+2z)(1-3z)(1+z+z^2) \rangle (1 + \alpha_{I_1} \lambda_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle)$$

$$- \lambda_{I_1}^2 \langle (1+z)(1+2z)(1-3z) \rangle),$$

$$h_{23} = \alpha_{I_1} \lambda_{I_1}^2 \langle (1+z)(1-3z)(1+2z+4z^2) \rangle,$$

$$h_{31} = \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle (1 + \alpha_{I_1} \lambda_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle),$$

$$h_{32} = \alpha_{I_2} \langle (1+z)(1+2z)(1-3z+z^2) \rangle,$$

$$h_{33} = -\alpha_{I_1} \lambda_{I_1}^2 \langle (1+z)(1+2z)(1-3z) \rangle.$$

Before finishing this example, let us show that the \mathcal{A} -module \mathcal{T}_P is *not* free. We show it by contradiction. Suppose that \mathcal{T}_P is free. Then the \mathcal{A} -module $M_r(T)$ is also free. Since the matrix D , a part of T , is nonsingular, the rank of $M_r(T)$ is m . Let V be a matrix in $(\mathcal{A})_m$ whose rows are m distinct generators of $M_r(T)$. As in (5.5), we have matrices $\tilde{Y}, \tilde{X}, N', D'$ over \mathcal{A} such that

$$\begin{bmatrix} N^t & D^t \end{bmatrix}^t = \begin{bmatrix} N^t & D^t \end{bmatrix}^t V, \quad \begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix} \begin{bmatrix} N^t & D^t \end{bmatrix}^t = V, \quad \tilde{Y}' N' + \tilde{X}' D' = E_1.$$

However the last matrix equation is inconsistent with the fact that the plant P does not have coprime factorization. Therefore \mathcal{T}_P is not free. Nevertheless we note that \mathcal{T}_P is projective by Theorem 3.3. \square

Example 5.3. (Continued) Let us follow part (b) in the proof of Theorem 3.3. Suppose that (i) of Theorem 3.3 holds, that is, the modules \mathcal{T}_P and \mathcal{W}_P are projective.

Consider again the plant P of (3.2). Let $F_1 = \{\lambda_{I_1}, \lambda_{I_2}\}$, where λ_{I_1} and λ_{I_2} are given as in (3.8). Then we have known that $\sum_{f \in F_1} f = 1$ and that there exists a right-coprime factorization of the plant over \mathcal{A}_f for every $f \in F_1$. By Lemma 4.2, the transposed plant P^t is stabilizable. We can construct its stabilizing controller by analogy to Example 5.2. Further we see that for both λ_{I_1} and λ_{I_2} , the transposed plant P^t has right-coprime factorizations over $\mathcal{A}_{\lambda_{I_1}}$ and $\mathcal{A}_{\lambda_{I_2}}$; that is, P has left-coprime factorizations over $\mathcal{A}_{\lambda_{I_1}}$ and $\mathcal{A}_{\lambda_{I_2}}$. Thus let $F_2 = \{\lambda_{I_1}, \lambda_{I_2}\}$. For $\lambda_{I_1} \in F_2$, we have the matrices $\tilde{N}_{I_1}, \tilde{D}_{I_1}, Y_{I_1}, X_{I_1}$ over $\mathcal{A}_{\lambda_{I_1}}$ such that $\tilde{N}_{I_1} Y_{I_1} + \tilde{N}_{I_1} X_{I_1} = E_2$ and

$$\begin{aligned} \tilde{N}_{I_1} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Y_{I_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad X_{I_1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ \tilde{D}_{I_1} &= \begin{bmatrix} \lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1+2z)(1-3z) \rangle & 0 \\ \lambda_{I_1}^{-1} \alpha_{I_1} \langle (1+z)(1-3z)(1+2z+4z^2) \rangle & 1 \end{bmatrix}. \end{aligned}$$

On the other hand, for $\lambda_{I_2} \in F_2$, we have the matrices $\tilde{N}_{I_2}, \tilde{D}_{I_2}, Y_{I_2}, X_{I_2}$ over $\mathcal{A}_{\lambda_{I_2}}$ such that $\tilde{N}_{I_2}Y_{I_2} + \tilde{D}_{I_2}X_{I_2} = E_2$ and

$$\begin{aligned} \tilde{N}_{I_2} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Y_{I_2} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad X_{I_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ \tilde{D}_{I_2} &= \begin{bmatrix} 0 & \lambda_{I_2}^{-1} \alpha_{I_2} \langle (1+z)(1+2z)(1-3z+z^2) \rangle \\ 1 & -\lambda_{I_2}^{-1} \alpha_{I_2} \langle (1+2z)(1+z+z^2)(1-3z+z^2) \rangle \end{bmatrix}. \end{aligned}$$

Now we let $F = \{\lambda_{I_1}^2, \lambda_{I_1}\lambda_{I_2}, \lambda_{I_2}^2\} (= \{f_1f_2 \mid f_1 \in F_1, f_2 \in F_2\})$. Then F still generates \mathcal{A} since $\lambda_{I_1}^2 + 2\lambda_{I_1}\lambda_{I_2} + \lambda_{I_2}^2 = 1$.

In the following we consider the case $f = \lambda_{I_1}^2$. Then using the matrix K of (4.3), we have (5.1) with $\nu = 1$, $K_f = K$ and $V_f = \lambda_{I_1}\Delta_{I_1}T$.

Then the ideal $I_{m\mathcal{A}}(f^\nu K_f)$ is generated by

$$\lambda_{I_1}^3, \quad \alpha_{I_1}\lambda_{I_1}^2 \langle (1+z)(1-3z)(1+2z+4z^2) \rangle, \quad \alpha_{I_1}\lambda_{I_1}^2 \langle (1+z)(1+2z)(1-3z) \rangle.$$

Thus since each of them is in Λ_{PI_1} , (5.2) holds. Further we can observe that for any integer ξ greater than 1, (5.3) holds since $\lambda_{I_1}^3 \in I_{m\mathcal{A}}(f^\nu K_f)$.

For the other cases $f = \lambda_{I_1}\lambda_{I_2}$ and $f = \lambda_{I_2}^2$, we can follow the relations of (5.2) and (5.3) analogously. Details are left to interested readers. \square

Remark 5.4. Since Anantharam's example in [1] is artificial, we do not present here the construction of a stabilizing controller. However we can construct it as part (c) in the proof of Theorem 3.3 (Since Anantharam in [1] did not consider the causality, we let $\mathcal{Z} = \{0\}$ so that $\mathcal{P} = \mathcal{F}$).

6. Causality of Stabilizing Controllers. In this section, we present two facts: (i) for a stabilizable causal plant, there exists at least one stabilizing causal controller and (ii) the stabilizing controller of the strictly causal plant is causal, which inherits Theorem 4.1 in § III of [14, p.888] and Proposition 1 of [11].

PROPOSITION 6.1. *For every stabilizable causal plant, there exists at least one stabilizing causal controller of the plant.*

Proof. In the construction of the stabilizing controller in part (c) of the proof of Theorem 3.3, the denominator matrix of (5.6) is \mathcal{Z} -nonsingular. Suppose that the obtained stabilizing controller is expressed as $\tilde{B}^{-1}\tilde{A}$ with the matrices \tilde{A} and \tilde{B} over \mathcal{A} such that \tilde{B} is \mathcal{Z} -nonsingular. Then since the relation $\tilde{B}^{-1}\tilde{A} = (\det(\tilde{B})E_m)^{-1}(\text{adj}(\tilde{B})\tilde{A})$ holds, every entry of $\tilde{B}^{-1}\tilde{A}$ is causal. \square

PROPOSITION 6.2. *For every stabilizable strictly causal plant, all stabilizing controllers of the plant must be causal.*

Proof. Suppose that the plant P is stabilizable and strictly causal. Suppose further that C is a stabilizing controller of P . We employ the notation from part (c) of the proof of Theorem 3.3. Thus, $a_{I_0}, \lambda_{I_0} \in \mathcal{A} \setminus \mathcal{Z}$ and $\tilde{Y}_{I_0}N_{I_0} + \tilde{X}_{I_0}D_{I_0} = E_m$ with $P = N_{I_0}D_{I_0}^{-1} \in \mathcal{F}(\mathcal{A}_{\lambda_{I_0}})$ from (5.5). Let $\mathcal{Z}_{\lambda_{I_0}} = \{z/1 \cdot u \in \mathcal{A}_{\lambda_{I_0}} \mid z \in \mathcal{Z}, u \text{ is a unit of } \mathcal{A}_{\lambda_{I_0}}\}$. Then this $\mathcal{Z}_{\lambda_{I_0}}$ is again a principal ideal of $\mathcal{A}_{\lambda_{I_0}}$.

Observe here that Lemma 8.3.2 of [13] and its proof hold even over a general commutative ring. According to its proof, there exist matrices \tilde{A} and \tilde{B} over $\mathcal{A}_{\lambda_{I_0}}$ such that $C = \tilde{B}^{-1}\tilde{A}$ and $\tilde{A}N_{I_0} + \tilde{B}D_{I_0} = E_m$ (\tilde{A} and \tilde{B} correspond to T and S , respectively, in the proof of Lemma 8.3.2 of [13]). Observe also that every entry of N_{I_0} is in $\mathcal{Z}_{\lambda_{I_0}}$. Thus reviewing the proof of Lemma 3.5 of [14], in which the calligraphic H and K in [14] correspond to $\mathcal{A}_{\lambda_{I_0}}$ and $\mathcal{Z}_{\lambda_{I_0}}$, respectively, we have $\det \tilde{B} \in \mathcal{A}_{\lambda_{I_0}} \setminus \mathcal{Z}_{\lambda_{I_0}}$. This implies that $\tilde{B}^{-1}\tilde{A} \in \mathcal{P}^{m \times n}$ by noting that $\lambda_{I_0} \in \mathcal{A} \setminus \mathcal{Z}$. Thus C is causal. \square

7. Further Work. In this paper we have presented criteria for feedback stabilizability. We have also presented a construction of a stabilizing controller to which Sule's method cannot be applied. Recently the first author[7] has developed a parameterization of stabilizing controllers, which is based on the results of this paper and which does not require coprime

factorizability. This can be applied to models to which Youla-Kučera parameterization cannot be applied.

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